for Φ, in the first approximation we obtain

$$b = \pm d, \quad 2(bn - dk)\psi_{\xi\eta} + (\varkappa + 1)bd^2\psi_\xi\psi_{\xi\xi} = \psi_{yy} + \psi_{zz} \qquad (2.10)$$

$$2(bn - dk)\frac{\partial\xi_0}{\partial\eta} + \left(\frac{\partial\xi_0}{\partial y}\right)^2 + \left(\frac{\partial\xi_0}{\partial z}\right)^2 = \frac{\varkappa+1}{2}bd^2(\psi_\xi + \psi_\xi^*), \quad \psi = \psi^*$$

For conditions at the shock front we use functions for $\xi = \xi_0$. The condition of impenetrability at the surface (2.8) for $y = y_0$ $(b \neq 0)$ is $b\partial f_0 / \partial\xi = \psi_y$. By specifying for the perturbation front originating in a quiescent gas $(\psi_\xi = \psi_\xi^* = 0)$ the form of circle $x^2 + y^2 = t^2$ (or of sphere $x^2 + y^2 + z^2 = t^2$), we find that $n = \mp k$, hence $\xi = d(x \pm t)$ and $\eta = k(x \mp t)$.

Thus for deriving the nonlinear equations for small unstable two- and three-dimensional perturbations of a sonic stream or in a quiescent gas it is necessary to use the characteristic variables of the related linear equations of one-dimensional flows. Although equations in terms of other variables can evidently be used, care must be taken to interpret these correctly. In particular, they can be used for defining flows whose unsteadiness becomes apparent only in the second approximation. Note that all solutions of the transonic equation in variables x and t [1] can be rewritten for Eqs. (2.2) and (2.10), by reducing these beforehand to the form appearing in [1]. This applies also to transformations that do not alter the form of the transonic equation (e.g. of that appearing in [4]) as well as the form of conditions at the shock front (or at a characteristic). Finally, a theorem of uniqueness, similar to that in [4] can be formulated for these equations.

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AXISYMMETRIC CONTACT PROBLEM ON THE IMPRESSION OF AN ELASTIC CYLINDER INTO AN ELASTIC LAYER

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A generalization is given of the problem on the impression of a circular stamp when the elastic stamp makes contact with an unbounded elastic layer. Application of the Hankel integral transform in the region of the layer and the properties of generalized orthogonality of eigenfunctions in the region of the circular cylinder (stamp) permits reducing the problem to an infinite system of

linear algebraic equations admitting of effective solution by the truncation method.

The classical problem of a rigid stamp impressed into an elastic half-space has been subjected to generalization in several directions in recent decades. Thus, the impression of a rigid stamp into an elastic layer has been considered in a number of papers (see [1], for example), on the other hand, the problem of the contact between an elastic cylinder and a half-space has been studied in [2]. Finally, the even more general problem on the impression of an elastic cylinder into an elastic layer has been investigated in [3]. The problem has been reduced to infinite linear algebraic systems, which are effectively solvable for sufficiently thick layers, since the solution is expanded in power series of a small parameter, the ratio of the cylinder radius to the layer thickness. The same problem is reduced in this paper to infinite systems of a different kind, which are suitable for any ratios between the geometric parameters. Results of numerical computations of the stiffness of the system under consideration are presented.

1. Formulation of the problem. Let us consider an axisymmetric problem on the frictionless impression of an elastic cylinder of radius a and height L in an elastic layer of thickness H which is at rest on a fixed base (Fig. 1). We introduce the cylindrical coordinates r, φ, z and assume that the upper endface of the cylinder is translated by a quantity w_0 and the surfaces $z = L$ and $z = -H$ are under smooth contact conditions. (The rest of the body surfaces remain stress free).

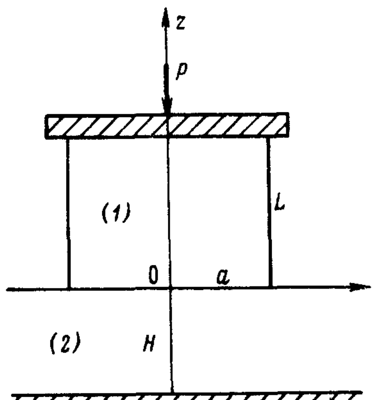

Fig. 1

It is convenient to go over to dimensionless coordinates by referring all the linear dimensions to the radius

$$\rho = \frac{r}{a}, \quad x = \frac{z}{a}, \quad l = \frac{L}{a}, \quad h = \frac{H}{a}$$

Marking quantities referring to the cylinder with the subscript 1, and those referring to the layer with the subscript 2, we obtain the boundary conditions of the problem and the matching conditions in the following form:

$$\sigma_\rho^{(1)}(1, x) = \tau_{x\rho}^{(1)}(1, x) = 0, \quad 0 \leqslant x \leqslant l \tag{1.1}$$

$$w^{(1)}(\rho, l) = w_* = \frac{w_0}{a}, \quad \tau_{x\rho}^{(1)}(\rho, l) = 0, \quad 0 \leqslant \rho \leqslant 1 \tag{1.2}$$

$$\tau_{x\rho}^{(2)}(\rho, -h) = w^{(2)}(\rho, -h) = 0, \quad 0 \leqslant \rho < \infty \tag{1.3}$$

$$\tau_{x\rho}^{(2)}(\rho, 0) = 0, \quad 0 \leqslant \rho < \infty \tag{1.4}$$

$$\sigma_x^{(2)}(\rho, 0) = 0, \quad 1 < \rho < \infty \tag{1.5}$$

$$\tau_{x\rho}^{(1)}(\rho, 0) = 0, \quad 0 \leqslant \rho \leqslant 1 \tag{1.6}$$

$$\sigma_x^{(1)}(\rho, 0) = \sigma_x^{(2)}(\rho, 0), \quad 0 \leqslant \rho \leqslant 1 \tag{1.7}$$

$$w^{(1)}(\rho, 0) = w^{(2)}(\rho, 0), \quad 0 \leqslant \rho \leqslant 1 \tag{1.8}$$

Here $(u, 0, w)$ are the displacement vector components, σ_x, σ_ρ, $\tau_{x\rho}$ are the stress tensor components. The notation $\nu_{1,2}$ and $G_{1,2}$ for the Poisson's ratio and the shear modulus are also used henceforth.

2. Reduction of the solution to an infinite algebraic system. The solution for a cylinder is constructed in the form of a series in homogeneous solutions of the problem of deformation of a cylinder with a free side surface for which the following complex functions are introduced [4, 5]:

$$\varepsilon_k(\rho) = J_0(p_k)J_0(p_k\rho) + \rho J_1(p_k)J_1(p_k\rho)$$
$$\delta_k(\rho) = \frac{2(1-\nu_1)}{p_k} J_1(p_k) J_0(p_k\rho)$$

which have the property of generalized Schiff orthogonality

$$\int_0^1 [\varepsilon_k'(\rho)\,\delta_n'(\rho) + \varepsilon_n'(\rho)\,\delta_k'(\rho)]\,\rho d\rho = \begin{cases} 0, & p_n^2 \neq p_k^2 \\ -2N_n, & p_n^2 = p_k^2 \end{cases} \tag{2.1}$$

Here p_k are roots of the transcendental equation

$$[2(1-\nu_1) - p^2]J_1^2(p) = p^2 J_0^2(p) \tag{2.2}$$

where two pairs of complex-conjugate roots of the form

$$p_k = \pm a_k \pm ib_k, \quad k = 1, 2, \ldots \quad (a_k, b_k > 0)$$

can be set in correspondence with each number k (the elementary solution of the deformation of a cylinder corresponds to the root $p_0 = 0$). As can be verified directly, in this problem it is sufficient to examine just roots in the half-plane $\operatorname{Re} p > 0$, hence, it will later be assumed that

$$p_k = a_k + ib_k, \; p_{-k} = a_k - ib_k, \quad k = 1, 2, \ldots$$

Thus the displacements and stresses in the cylinder can be sought as series satisfying the conditions (1.1), (1.2) (the sign of the summation Σ' is extended over all integer k except zero; the argument ρ in the functions ε_k, δ_k and their derivatives ε_k', δ_k' is omitted)

$$w^{(1)}(\rho, x) = w_* + \frac{\sigma_0(x-l)}{2G_1(1+\nu_1)} - \sum_k{}' C_k(\varepsilon_k - \delta_k)\frac{\operatorname{sh} p_k(l-x)}{\operatorname{ch} p_k l} \tag{2.3}$$
$$u^{(1)}(\rho, x) = -\frac{\sigma_0\nu_1\rho}{2G_1(1+\nu_1)} + \sum_k{}' C_k(\varepsilon_k' + \delta_k')\frac{\operatorname{ch} p_k(l-x)}{\operatorname{ch} p_k l}$$
$$\sigma_x^{(1)}(\rho, x) = \sigma_0 - 2G_1\sum_k{}' C_k\frac{(\rho\varepsilon_k')'}{\rho}\frac{\operatorname{ch} p_k(l-x)}{\operatorname{ch} p_k l}$$
$$\tau_{x\rho}^{(1)}(\rho, x) = -2G_1\sum_k{}' C_k\varepsilon_k' p_k\frac{\operatorname{sh} p_k(l-x)}{\operatorname{ch} p_k l}$$
$$\sigma_\rho^{(1)}(\rho, x) = 2G_1\sum_k{}' C_k\left(\varepsilon_k'' + \delta_k'' - \frac{\nu_1 p_k^2\delta_k}{1-\nu_1}\right)\frac{\operatorname{ch} p_k(l-x)}{\operatorname{ch} p_k l}$$

The terms corresponding to the numbers k and $-k$ in the formulas presented are complex-conjugates ($\varepsilon_k = \varepsilon_{-k}$, etc.).

The solution for a layer can be constructed as a Hankel integral [1]. Let us present here just the expression for the normal stress and displacement at points of the surface $x = 0$

$$\sigma_x^{(2)}(\rho, 0) = \int_0^\infty \lambda A(\lambda)\,J_0(\lambda\rho)\,d\lambda \tag{2.4}$$

$$w^{(2)}(\rho, 0) = \frac{1-\nu_2}{G_2}\int_0^\infty A(\lambda) f(\lambda) J_0(\lambda\rho)\, d\lambda, \quad 0 \leqslant \rho < \infty \tag{2.5}$$

$$f(\lambda) = \frac{\operatorname{sh}^2 \lambda h}{\lambda h + \operatorname{sh}\lambda h \operatorname{ch}\lambda h}$$

(conditions (1.3) and (1.4) are satisfied for such a selection of the solution).

Conditions (1.5) and (1.7) can be written as

$$\sigma_x^{(2)}(\rho, 0) = \begin{cases} \sigma_x^{(1)}(\rho, 0), & 0 \leqslant \rho \leqslant 1 \\ 0, & 1 < \rho < \infty \end{cases}$$

Hence, recalling the third formula in (2.3) and (2.4), we can obtain

$$A(\lambda) = \int_0^1 \rho J_0(\lambda\rho)\, \sigma_x^{(1)}(\rho, 0)\, d\rho = \sigma_0\alpha_0(\lambda) + 2G_1 \sum_k{}' C_k \alpha_k(\lambda) \tag{2.6}$$

Here

$$\alpha_0(\lambda) = \int_0^1 \rho J_0(\lambda\rho)\, d\rho \tag{2.7}$$

$$\alpha_k(\lambda) = -\int_0^1 J_0(\lambda\rho)(\rho\varepsilon_k')'\, d\rho = -\lambda \int_0^1 J_1(\lambda\rho)\, \rho\varepsilon_k' d\rho, \quad k = \pm 1, \pm 2, \ldots$$

The unsatisfied conditions (1.6) and (1.8), which become

$$2G_1 \sum_k{}' C_k \varepsilon_k' p_k \operatorname{th} p_k l = 0, \quad 0 \leqslant \rho \leqslant 1 \tag{2.8}$$

$$w_* - \frac{\sigma_0 l}{2G_1(1+\nu_1)} - \sum_k{}' C_k(\varepsilon_k - \delta_k) p_k \operatorname{th} p_k l = w^{(2)}(\rho, 0), \quad 0 \leqslant \rho \leqslant 1$$

remain, where $w^{(2)}(\rho, 0)$ is determined in terms of the coefficients σ_0, C_k by means of (2.5) and (2.6).

The generalized orthogonality relation (2.1) permits reduction of the equality (2.8) to an infinite system by application of the following method. Multiplying (2.8), respectively, by the functions $\nu_1\rho^2 [2G_1(1+\nu_1)]^{-1}$ and ρ, subtracting them and integrating over the section [0, 1] with respect to ρ, we arrive after some computations at

$$\sigma_0 \frac{l}{4G_1(1+\nu_1)} + \int_0^1 w^{(2)}(\rho, 0)\, \rho d\rho = \frac{1}{2} w_*$$

Multiplying (2.8) analogously by $(\varepsilon_n' + \delta_n')\rho$ and $2G_1(\rho\varepsilon_n')'$, subtracting and integrating, we have

$$2C_n N_n p_n \operatorname{th} p_n l - \int_0^1 w^{(2)}(\rho, 0)(\rho\varepsilon_n')'\, d\rho = 0, \quad n = \pm 1, \pm 2, \ldots$$

Taking account of (2.5) and (2.6), we arrive at an infinite system of equations after some formal calculations:

$$\begin{cases} a_0 X_0 + g \sum_k b_{0k} X_k = \frac{1}{4} & \left(g = \frac{G_1}{G_2}\right) \\ a_n X_n + g \sum_k b_{nk} X_k = 0, & n = \pm 1, \pm 2, \ldots \end{cases} \tag{2.9}$$

Here

$$a_0 = \frac{l}{4(1+\nu_1)} \tag{2.10}$$

$$a_n = p_n N_n \operatorname{th} p_n l, \quad n = \pm 1, \pm 2, \ldots$$

$$b_{nk} = (1 - \nu_2) \int_0^\infty \alpha_n(\lambda)\, \alpha_k(\lambda)\, f(\lambda)\, d\lambda; \quad n, k = 0, \pm 1, \pm 2, \ldots$$

The Σ sign extends over all integer k; the quantities X_0, X_k are connected with the fundamental unknowns σ_0, C_k by the relations

$$X_0 = \frac{\sigma_0}{2G_1 w_*}, \quad X_k = \frac{C_k}{w_*}, \quad k = \pm 1, \pm 2, \ldots$$

Passage to the limit $h \to 0$ or $G_2 \to \infty$ yields

$$X_0 = a_0 / 4, \quad X_n = 0, \quad n = \pm 1, \pm 2, \ldots$$

This result determines the elementary solution of the problem of the compression of a cylinder by smooth rigid planes

$$\sigma_x^{(1)}(\rho, x) = 2G_1(1 + \nu_1) w_* / l$$

There hence results that the system (2.9) is solvable most effectively for small h (by the method of iteration for example).

3. Numerical solution. The roots of the characteristic equation (2.2) needed for the numerical realization of the solution found were taken from [6] for $\nu_1 = 0.3$; the Bessel functions of complex argument were determined by using the integral representation [7]

$$J_q(z) = \frac{(z/2)^q}{\Gamma(q + 1/2)\,\Gamma(1/2)} \int_{-1}^{1} e^{izt} (1 - t^2)^{q - 1/2}\, dt, \quad \operatorname{Re} q > -\frac{1}{2}$$

and the recursion formula

$$J_0(z) = \frac{2}{z} J_1(z) - J_2(z)$$

The quadratures (2.1) and (2.7) turn out to be evaluable explicitly

$$N_n = (1 - \nu_1) J_1^2(p_n) \left\{ 2 \frac{1 - \nu_1}{p_n^2} J_1(p_n) [J_1(p_n) - p_n J_0(p_n)] + J_0^2(p_n) \right\}$$

$$\alpha_0(\lambda) = \frac{1}{\lambda} J_1(\lambda)$$

$$\alpha_n(\lambda) = \frac{\lambda J_1(\lambda)}{\lambda^2 - p_n^2} \left[2(1 - \nu_1) J_1^2(p_n) - \frac{2\lambda^2}{\lambda^2 - p_n^2} p_n J_0(p_n) J_1(p_n) \right] +$$
$$\frac{2\lambda^2 J_0(\lambda)}{(\lambda^2 - p_n^2)^2} p_n^2 J_1^2(p_n), \quad n = \pm 1, \pm 2, \ldots$$

The complex coefficients b_{nk} in (2.10) for the infinite system (2.9) were determined by Simpson's rule. The metnod of reduction was used for the numerical solution on a BESM-4 computer.

The ratio between the total axial force P and the magnitude of the displacement of the cylinder upper endface w_0 can be called the stiffness of the elastic system considered. The following simple representation for the stiffness

$$C = P / w_0 = 2\pi G_1 a X_0$$

was obtained by integrating the stress σ_x in the third relationship in (2.3) over the contact area.

Table 1

$h \backslash l$	0	$^1/_{16}$	$^1/_8$	$^1/_4$	$^1/_2$	1	2	4	8
0	—	131	65.3	32.7	16.3	8.17	4.08	2.04	1.02
$^1/_4$	39.9	30.6	24.8	18.0	11.6	6.78	3.71	1.94	0.996
$^1/_2$	21.0	18.1	15.9	12.8	9.19	5.88	3.42	1.86	0.974
1	12.0	11.0	10.1	8.72	6.88	4.84	3.04	1.75	0.940
2	8.10	7.61	7.21	6.50	5.42	4.07	2.72	1.63	0.907
∞	5.44	5.23	5.02	4.66	4.08	3.27	2.34	1.49	0.860

Presented in Table 1 are values of the dimensionless quantity $C / G_1 a = 2\pi X_0$ computed for different values of the geometric parameters for $G_1 = G_2, \nu_1 = \nu_2 = 0.3$.

Let us note that the solution of the problem on the elastic contact between a cylinder and a layer which adheres to a fixed base can also be reduced to an infinite system of equations of the type (2.9) and (2.10), where just the function $f(\lambda)$ (see (2.5)) should be replaced in conformity with [8] in this case by

$$f(\lambda) = 1 - \frac{\mu(1+\mu) + 4(1-\nu_2)^2 - (3-4\nu_2)e^{-\mu}\operatorname{sh}\mu}{(3-4\nu_2)\operatorname{sh}^2\mu + \mu^2 + 4(1-\nu_2)^2}, \quad \mu = \lambda h$$

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